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Cauchy's Problem for a Singular Parabolic Partial Differential Equation

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I. INTRODUCTION

In recent years analytic function theory has been shown to play a basic role in the investigation of existence and uniqueness theorems for solutions to elliptic partial differential equations ([6], [8], [17]). An approach which has proved particularly fruitful is that of integral operators ([1], [12], [17]), from whose use complete families of solutions can be obtained, thus enabling one to construct the Bergman kernel function and solve the Dirichlet and Neumann problems ([2]). For singular elliptic equations serious difficulties arise due to the nonregularities in the kernels of such operators, as well as the failure of Green's representation to hold in a neighborhood of the singular curve. In such cases recourse is often made to the use of operators whose path of integration is a contour in the complex plane ([7], [15]). In this paper we apply integral operator techniques in conjunction with function theoretic methods to establish an existence, uniqueness, and representation theorem to Cauchy's problem for the singular parabolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\nu}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \tag{1}$$

where ν is a real parameter. Equation (1) has previously been studied by Bragg and Haimo ([4], [13]) and is sometimes known as the generalized heat equation.

The purpose of our work is twofold:

1. To the author's knowledge this is the first time the integral operator techniques described above have been applied to a parabolic equation.

2. It shows not only the sufficiency but also the necessity of relying on function theoretic methods in the investigation of singular initial value problems such as the one considered here.

Cauchy's problem for equation (1) can be formulated as follows: to find a unique function $u(x, t) \in C^2$ satisfying equation (1) for $0 < t < t_0$, $-\infty < x < \infty$, such that $u(x, t)$ continuously assumes given boundary data on the initial line $t = 0$. For $\nu > 0$ this problem has been solved by Cholewinski and Haimo ([5]) provided that for each arbitrarily small positive ϵ, $u(x, t)$ satisfies a bound of the form

$$|u(x, t)| \leqslant Me^{Ax^2}, \qquad 0 \leqslant t \leqslant t_0 - \epsilon \tag{2}$$

for positive constants $M = M(\epsilon)$, $A = A(\epsilon)$. The example $u(x, t) = t^{-\nu-\frac{1}{2}}e^{-x^2/4t}$ shows that such a condition is no longer sufficient to insure uniqueness (to say nothing of existence) if $\nu < -\frac{1}{2}$. By use of the fundamental solution to equation (1) Haimo has furthermore obtained expansion theorems in terms of a complete polynomial set for solutions of (1) which are analytic in a neighborhood of the origin, provided again that $\nu > 0$ ([14]). This work is a generalization of Widder's result for the case $\nu = 0$ ([18]) and is based on rather lengthy calculations involving Laguerre polynomials and Bessel functions ([13]). Here by the use of integral operator methods we obtain a somewhat weakened form of Haimo's result as a direct consequence of Widder's work, with the added advantage that the representation theorem is now valid for all real values of ν except $2\nu = -1, -2, -3,\ldots$. From now on the case $\nu = 0$ will always be excluded since in this case (1) becomes the heat equation for which the existence, uniqueness, and representation of solutions to Cauchy's problem are well-known results ([11], [16]).

At this point we make the important observation that if $u(x, t)$ is a solution of equation (1) which is analytic in a neighborhood of the origin, then, provided $2\nu \neq -1, -2, -3,\ldots$, $u(x, t)$ is an even function of x which is uniquely determined by its axial values $u(0, t)$. This follows from the fact that the line $x = 0$ is a singular curve of the regular type with indices 0 and $1 - 2\nu$ such that $\partial u/\partial x = 0$ along the axis $x = 0$ (cf. [10]).

II. Fractional Integration and the Generalized Heat Equation

From the relationship holding between the Riemann-Liouville operator of fractional integration

$$I_{x^2}^{\nu,\alpha} f(x) \stackrel{\text{def}}{=} \frac{2}{\Gamma(\alpha)} \int_0^1 (1 - \xi^2)^{\alpha-1} \xi^{2\nu+1} f(x\xi)\, d\xi; \qquad \alpha > 0, \qquad \nu > -\tfrac{1}{2} \tag{3}$$

and Bessel's differential operator

$$L_\nu f \overset{\text{def}}{=} \frac{d^2f}{dx} + \frac{2\nu}{x}\frac{df}{dx} \tag{4}$$

viz. ([10])

$$I_{x^2}^{\nu,\alpha} L_\nu f = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} f \tag{5}$$

we can relate twice continuously differentiable solutions $v(x, t)$ of the classical one dimensional heat equation which are even functions of x to solutions $u(x, t)$ of equation (1) by the relation

$$u(x, t) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu)\,\Gamma(\frac{1}{2})} \int_0^1 v(x\xi, t)(1 - \xi^2)^{\nu-1}\, d\xi \tag{6}$$

provided $\nu > 0$. If we assume further that $v(x, t)$ is analytic at the origin (and hence also $u(x, t)$) then, following Erdélyi ([9]), for x and t sufficiently small, we can rewrite (6) as the contour integral

$$\begin{aligned} u(x, t) &= \frac{\int_C v(x\xi, t)(1 - \xi^2)^{\nu-1} d\xi}{\int_C (1 - \xi^2)^{\nu-1} d\xi} \\ &\overset{\text{def}}{=} \Lambda v(x, t) \end{aligned} \tag{7}$$

where for $\nu > 0$ C is the interval $[0, 1]$ and for $\nu < 0$, $2\nu \neq -1, -3, -5,...$ C is a loop beginning and ending at $\xi = 0$ and encircling $\xi = 1$ counterclockwise. The operator Λ defined above can furthermore be inverted ([9]) to obtain

$$\begin{aligned} v(x, t) &= \frac{\int_C u(x\xi, t)\, \xi^{2\nu}(1 - \xi^2)^{-\nu-1} d\xi}{\int_C \xi^{2\nu}(1 - \xi^2)^{-\nu-1} d\xi} \\ &\overset{\text{def}}{=} \Lambda^{-1} u(x, t) \end{aligned} \tag{8}$$

where C is a loop starting and ending at $\xi = 0$ and encircling $\xi = 1$ if $\nu > 0$, and a loop starting and ending at $\xi = 1$ and encircling $\xi = 0$ if $\nu < 0$, $2\nu \neq -1, -3, -5,...$. In view of the unique dependence of $u(x, t)$ on its axial values for $2\nu \neq -1, -2, -3,...$, and the fact that from (7) we have $u(0, t) = v(0, t)$, it can be deduced from corollary 4.1 of [18] (which solves the Cauchy-Kowalewski boundary value problem for the heat equation "in the large") that if $2\nu \neq -1, -2, -3,...$, every solution of equation (1) which is analytic in a neighborhood of the origin can be uniquely expressed locally in the form of equation (7). If so desired, equations (7) and (8) can now be used for the purpose of analytic continuation.

In [16] Rosenbloom and Widder obtained a set of polynomial solutions to the one dimensional heat equation which are complete in the space of solutions analytic in some neighborhood of the origin ([18]). These are known as heat polynomials and can be expressed as

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n-2k)! k!} \tag{9}$$

This result was subsequently generalized by Bragg and Haimo to the case of equation (1) for $\nu > 0$ ([4], [13]), with the corresponding "generalized" heat polynomials given by

$$P_{n,\nu}(x, t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + n - k + \frac{1}{2})} x^{2n-2k} t^k \tag{10}$$

Observing that

1. $P_{n,\nu}(x, t)$ satisfies equation (1) not only for $\nu > 0$ but for all real values of ν,

2) $P_{n,\nu}(0, t) = 2^{2n} \dfrac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} t^n,$

3) $v_{2n}(0, t) = \dfrac{(2n)!}{n!} t^n,$

we have the immediate result (due to the unique dependence of solution to equation (1) on its axial values):

LEMMA 1. *For* $2\nu \neq -1, -2, -3, \ldots,$

$$A v_{2n}(x, t) = h_n^{\nu} P_{n,\nu}(x, t)$$

where

$$h_n^{\nu} = \frac{\Gamma(n + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(n + \nu + \frac{1}{2}) \Gamma(\frac{1}{2})}$$

In Lemma 1 use was made of Legendre's duplication formula to evaluate the constant h_n^{ν}. We note in passing that from the relationships ([13], [16])

$$v_{2n}(x, -t) = t^n H_{2n} \left(\frac{x}{(4t)^{\frac{1}{2}}} \right) \tag{11}$$

$$P_{n,\nu}(x, -t) = (-1)^n 2^{2n} n! t^n L_n^{\nu - \frac{1}{2}} \left(\frac{x^2}{4t} \right) \tag{12}$$

where H_{2n} denotes Hermite's polynomial and $L_n^{\nu-\frac{1}{2}}$ Laguerre's polynomial, the following formula, due originally to Uspensky, is an immediate consequence of Lemma 1:

$$\Gamma(n+\nu+1)\int_{-1}^{1}(1-\xi^2)^{\nu-\frac{1}{2}}H_{2n}(x^{\frac{1}{2}}\xi)\,d\xi = (-1)^n\pi^{\frac{1}{2}}(2n)!\,\Gamma(\nu+\tfrac{1}{2})L_n^{\nu}(x);$$
$$\nu > -\tfrac{1}{2} \qquad (13)$$

III. Existence, Uniqueness, and Representation of the Solution to Cauchy's Problem

Our first result in this section is to derive an expansion theorem for solutions of equation (1) which are analytic at the origin in terms of generalized heat polynomials. Using different methods, the following theorem has been obtained in a somewhat stronger form by Haimo ([13], [14]) for the case $\nu > 0$.

Theorem 1. *Let $u(x, t)$ be a solution of equation* (1) *which is analytic for $|t| < \sigma$, $|x| < \sigma$, and assume $2\nu \neq -1, -2, -3,\ldots$. Then $u(x, t)$ can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$, and expanded in a series of the form*

$$u(x,t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x,t),$$

the series converging pointwise for $0 < t < \sigma$, $-\infty < x < \infty$.

Proof. Let $v(x, t) = \Lambda^{-1}u(x, t)$. Then $v(x, t)$ is a solution of the heat equation, even with respect to x, and analytic for $|t| < \sigma$, $|x| < \sigma$. Hence from [16] and [18] $v(x, t)$ is analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$, and we can write

$$v(x,t) = \sum_{n=0}^{\infty} b_n v_{2n}(x,t) \qquad (14)$$

where the series converges uniformly for each fixed t, $0 < t < \sigma$, x contained in any compact region of the complex x plane. Hence $u(x, t) = \Lambda v(x, t)$ is analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$, and for $0 < t < \sigma$ we can apply the operator termwise in (14) and use Lemma 1 to obtain

$$u(x,t) = \sum_{n=0}^{\infty} b_n h_n^{\nu} P_{n,\nu}(x,t) \qquad (15)$$

for $0 < t < \sigma$, $-\infty < x < \infty$. Setting $b_n h_n^\nu$ equal to the new constant a_n establishes the theorem. (If $\nu > 0$ we could now use the results of [13] to show (15) in fact converges absolutely in the strip $|t| < \sigma$, $-\infty < x < \infty$, and represents $u(x, t)$ there. However for our purposes this is not required.)

We now proceed to the main result of this paper, i.e. the solution of Cauchy's problem for the generalized heat equation. We first require a preliminary definition.

DEFINITION 1. An entire function of a complex variable is of growth (ρ, τ) if and only if it is of order less than or equal to ρ, and is of type τ if of order ρ.

From [3] we have the result that the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of growth (ρ, τ) if and only if

$$\limsup_{n\to\infty} \frac{n}{e\rho} \mid a_n \mid^{\rho/n} \leqslant \tau. \tag{16}$$

Note that if $f(z)$ is an entire function of growth (ρ, τ), then $f(z^2)$ is an even entire function of growth $(2\rho, \tau)$.

THEOREM 2. *If $2\nu \neq -1, -2, -3, \ldots$, then there exists a unique solution to Cauchy's problem for equation* (1) *which is of class C^2 for $0 < t < \sigma$, $-\infty < x < \infty$, provided that*

1. *$u(x, t)$ is analytic for $|x| < \sigma$, $|t| < \sigma$, and*

2. *$u(x, 0) = g(x)$ where $g(x)$ is the restriction to the real axis of an even entire function of growth $(2, \frac{1}{4}\sigma)$.*

If $g(x)$ is represented by its Taylor series

$$g(x) = \sum_{n=0}^{\infty} a_n x^{2n}$$

then for $0 < t < \sigma$, $-\infty < x < \infty$, $u(x, t)$ has the representation

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$$

Proof. Let

$$f(x) = \Lambda^{-1} g(x) = \sum_{n=0}^{\infty} a_n \Lambda^{-1} x^{2n}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu + n + \frac{1}{2})}{\Gamma(n + \frac{1}{2})\,\Gamma(\nu + \frac{1}{2})} x^{2n},$$

termwise integration being permissible since the Taylor series for $g(x)$ converges uniformly in the complex x plane. As is easily seen from its series development, $f(x)$ is an entire function of growth $(2, \frac{1}{4}\sigma)$ and hence from [16] and [18] we can construct a unique solution of the heat equation, $v(x, t)$, such that $v(x, t)$ is analytic for $|t| < \sigma$, $-\infty < x < \infty$ and $v(x, 0) = f(x)$. This function is given explicitly by

$$v(x, t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\nu + n + \frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})\,\Gamma(\nu + \frac{1}{2})} v_{2n}(x, t) \tag{17}$$

Let $u(x, t) = \Lambda v(x, t)$. Then $u(x, t)$ is a solution of the generalized heat equation which is analytic for $|t| < \sigma$, $-\infty < x < \infty$. Furthermore

$$u(x, 0) = \Lambda x(v, 0) = \Lambda f(x) = \sum_{n=0}^{\infty} a_n \Lambda \Lambda^{-1} x^{2n} = \sum_{n=0}^{\infty} a_n x^{2n} = g(x), \tag{18}$$

termwise integration being permissible due to the uniform convergence of the Taylor series for $f(x)$. Applying the operator Λ termwise in equation (17) for $0 < t < \sigma$ as was done in Theorem 1 yields the representation

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t). \tag{19}$$

The solution $u(x, t)$ is uniquely determined since if a second solution $u_1(x, t)$ existed satisfying conditions (1) and (2) of the theorem, then $w(x, t) = u(x, t) - u_1(x, t)$ would also be a solution of equation (1) satisfying the hypothesis of Theorem 2 with $g(x) = 0$, and hence $v(x, t) = \Lambda^{-1} w(x, t)$ would be a solution of the heat equation analytic at the origin which vanishes along the characteristic $t = 0$. From [18] $v(x, t)$ can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$, and must be identically zero there, which implies $\Lambda v(x, t) = w(x, t) \equiv 0$, i.e., $u(x, t) = u_1(x, t)$ in a neighborhood of the origin and both can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$. This shows that the solution $u(x, t)$ we have constructed is unique in the class of solutions analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$. But from the analytic theory of parabolic partial differential equations ([11]) we have that any solution which is of class C^2 in a domain not containing the singular line $x = 0$ must be an analytic function of x for each fixed t and hence if it is analytic for $|t| < \sigma$, $|x| < \sigma$, agrees with the above constructed solution $u(x, t)$. The theorem is now completely proved.

Recall from part I that the example $u(x, t) = t^{-\nu-\frac{1}{2}} e^{-x^2/4t}$ shows that in order to assure uniqueness, condition (1) of Theorem 2 is a necessary as well as a sufficient hypothesis for the case $\nu < -\frac{1}{2}$, even if an *a priori* bound of the form of equation (2) is assumed. Condition 2 now appears as a kind of "compatibility" condition, since the proof of Theorem 2 shows that any solution satisfying the first condition automatically has an analytic continuation into the strip $|t| < \sigma$, $-\infty < x < \infty$, under which $u(x, 0)$ is continued to an even entire function of growth $(2, \frac{1}{4}\sigma)$.

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